

# Bounds on the Quenched Pressure and Main Eigenvalue of the Ruelle Operator for Brownian Type Potentials

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## Abstract

In this paper we consider a random potential derived from the Brownian motion. We obtain upper and lower bounds for the expected value of the main eigenvalue of the associated Ruelle operator and for its quenched topological pressure. We also exhibit an isomorphism between the space  $C(\Omega)$  endowed with its standard norm and a proper closed subspace of the Skorokhod space which is used to obtain a stochastic functional equation for the main eigenvalue and for its associated eigenfunction.

**Key-words:** Brownian motion, One-dimensional lattice, Quenched Pressure, Random potentials, Ruelle operator, Thermodynamic formalism.

**MSC2010:** 37A50, 37A60, 60J65, 82B05, 97K50.

## 1 Introduction

The basic idea of the Ruelle operator is based on the transfer matrix method introduced by Kramers and Wannier and independently by Montroll to explicitly compute partition functions in Statistical Mechanics. This operator was introduced in 1968 by David Ruelle [Rue68] and was used to describe local to global properties of systems with infinite range interactions. Among other things, Ruelle used this operator to prove the existence of a unique DLR-Gibbs measure for a lattice gas system depending on infinitely many coordinates. The acronym DLR stands for Dobrushin, Lanford and Ruelle.

With the advent of the Markov partitions due to Y. Sinai, remarkable applications of this operator to Hyperbolic dynamical systems on compact

manifolds were further obtained by Ruelle, Sinai and Bowen, see [Rue68, Sin72, Bow08]. Since its creation, this operator remains to have a great influence in many fields of pure and applied Mathematics. It is a powerful tool to study topological dynamics, invariant measures for Anosov flows, statistical mechanics in one-dimension, meromorphy of the Zelberg and Ruelle dynamical zeta functions, multifractal analysis, conformal dynamics in one dimension and fractal dimensions of horseshoes, just to name a few. For those topics, we recommend, respectively, [Bar11, Bow08, Bow79, Mañ90, MM83, PP90, Rue02, Sin72] and references therein.

In order to describe our contributions, we shall start by briefly recalling the classical setting of the Ruelle operator. A comprehensive exposition of this subject can be found in [Bal00, Bow08, PP90].

We now introduce the state space and the dynamics that will guide the following discussions. In this paper, the state space will be the Bernoulli space  $\Omega = \{0, 1\}^{\mathbb{N}}$  and the dynamics is given by  $\sigma : \Omega \rightarrow \Omega$ , the left shift map. When topological concept are concerned, the state space  $\Omega$  is regarded as a metric space where the distance between points  $x$  and  $y \in \Omega$  is given by

$$d(x, y) = \frac{1}{2^N}, \text{ where } N = \inf\{i \in \mathbb{N} : x_i \neq y_i\}.$$

We shall remark that  $\Omega$  is a compact metric space when endowed with this metric. All of the results presented in this paper can be naturally extended to  $\{0, 1, \dots, m-1\}^{\mathbb{N}}$ ,  $m \geq 2$ , but for the sake of simplicity, we restrain ourselves to the binary case. A summary of the changes required to study this more general case are presented in Section 7.

Back to the case  $\Omega = \{0, 1\}^{\mathbb{N}}$ , the set of all real continuous functions defined over  $\Omega$  will be denoted by  $C(\Omega)$  and the Ruelle transfer operator, or simply the Ruelle operator, associated to a continuous function  $f : \Omega \rightarrow \mathbb{R}$  will be denoted by  $\mathcal{L}_f$ . This operator acts on the space of all continuous functions sending  $\varphi$  to  $\mathcal{L}_f(\varphi)$  which is defined for any  $x \in \Omega$  by the following expression

$$\mathcal{L}_f(\varphi)(x) = \sum_{a=0,1} \exp(f(ax))\varphi(ax),$$

where  $ax \equiv (a, x_1, x_2, \dots)$ . The function  $f$  is usually called **potential**.

In this work a major role will be played by  $C^\gamma(\Omega)$ , the space of  $\gamma$ -Hölder continuous functions, with the Hölder exponent  $\gamma$  satisfying  $0 < \gamma < 1/2$ . This space is the set of all functions  $f : \Omega \rightarrow \mathbb{R}$  satisfying

$$\text{Hol}(f) = \sup_{x, y \in \Omega: x \neq y} \frac{|f(x) - f(y)|}{d(x, y)^\gamma} < +\infty.$$

Equipped with its standard norm  $\|\cdot\|_\gamma$  given by  $C^\gamma(\Omega) \ni f \mapsto \|f\|_\gamma \equiv \|f\|_\infty + \text{Hol}(f)$ , the space  $C^\gamma(\Omega)$  is a Banach space, see [PP90] for more details.

Furthermore, we present the suitable version of the Ruelle-Perron-Fröbenius Theorem that will be required for our discussions. See [Bal00, Bow08, PP90] for a proof.

**Theorem A** (Ruelle-Perron-Fröbenius, RPF). *If  $f$  is a potential in  $C^\gamma(\Omega)$  for some  $0 < \gamma < 1$ , then  $\mathcal{L}_f : C^\gamma(\Omega) \rightarrow C^\gamma(\Omega)$  have a simple positive eigenvalue of maximal modulus  $\lambda_f$  with a corresponding strictly positive eigenfunction  $h_f$  and a unique Borel probability measure  $\nu_f$  on  $\Omega$  such that,*

- i) the remainder of the spectrum of  $\mathcal{L}_f : C^\gamma(\Omega) \rightarrow C^\gamma(\Omega)$  is contained in a disc of radius strictly smaller than  $\lambda_f$ ;*
- ii) the probability measure  $\nu_f$  satisfies  $\mathcal{L}_f^*(\nu_f) = \lambda_f \nu_f$ , where  $\mathcal{L}_f^*$  denotes the dual of the Ruelle operator;*
- iii) for all continuous functions  $\varphi \in C(\Omega)$  we have*

$$\lim_{n \rightarrow \infty} \left\| \frac{\mathcal{L}_f^n(\varphi)}{\lambda_f^n} - h_f \int_\Omega \varphi d\nu_f \right\|_\infty = 0.$$

Whereas in the classical theory of the Ruelle operator,  $f$  is some fixed Hölder, Walters [Wal07, Bou01] or Bowen [Bow08, Wal01] potential, here we are interested in obtaining almost certain results for random potentials.

**Definition 1** (Random Potentials). *Let  $(\Psi, \mathcal{F}, \mathbb{P})$  be a complete probability space. A random potential is a map  $X : \Psi \times \Omega \rightarrow \mathbb{R}$  satisfying for any fixed  $x \in \Omega$  that  $\psi \mapsto X(\psi, x)$  is  $\mathcal{F}$ -measurable. A random potential  $X$  is said to be a **continuous random potential** if  $x \mapsto X(\psi, x) \in C(\Omega)$  for  $\mathbb{P}$ -almost all  $\psi \in \Psi$ .*

Although some of our results could be stated for a large class of random potentials, we focus on the random potentials of Brownian type (see the next section for the definition of such potentials).

When moving into the realm of random potentials the basic theory of the Thermodynamic Formalism needs to be reconstructed from scratch. In [FS88] random potentials were considered and this is the setting of our work. Latter, more general settings were introduced. The theory was extended to cover countable random Markov subshifts of finite type, allowing randomness to be considered also in the adjacency matrix, we refer the reader to [BG95,

Bog96, DKS08, FS88, GK00, Kif08, KL06, MSU11], for this more general setting.

As mentioned above the randomness here is considered exclusively in the potential. Since this is simpler setting, for the reader's convenience we proved here, in full details for the Brownian potential, that important quantities such as the spectral radius, the topological pressure and the main eigenvalue of the Ruelle operator can be associated to random variables with finite first moment in Wiener spaces. At the end of Section 2 we explain how to obtain similar results for Brownian potential using the general setting considered in [Kif08, KL06].

Working with random potentials, we are led to consider questions similar in spirit to the ones we ask when studying disordered systems in Statistical Mechanics, for instance the existence of annealed and quenched free energy and pressure, respectively, see [MPV87]. Our approach provides a new way to obtain the existence and finiteness of quenched topological pressure defined by

$$P^{\text{quenched}}(X) \equiv \mathbb{E} \left[ \sup_{\mu \in \mathcal{P}_\sigma(\Omega)} \left\{ h(\mu) + \int_{\Omega} X d\mu \right\} \right],$$

(note that the existence of the rhs above is itself a deep question) as well as the identity  $P^{\text{quenched}}(X) = \mathbb{E} [\lim_{n \rightarrow \infty} n^{-1} \log \mathcal{L}_X^n(1)(\sigma^n(x))], \forall x \in \Omega$  for a large class of continuous random potential  $X$  not invoking the classical replica trick. On the other hand, despite the progress achieved with this method, it has the disadvantage of not being able to exhibit explicitly these quantities.

Our main results are Theorems 1, 2, the expression (3), and the bounds for the quenched pressure in Section 6.

The paper is organized as follows. In Section 2 we introduce the so-called Brownian potential and we prove that the main eigenvalue for the random Ruelle operator associated to the Brownian potential almost surely exists. We also prove in this section that phase transition for this random potential, in the sense of the Definition 2, is almost surely absent. In Section 3 and 4 we prove a Banach isomorphism between  $C(\Omega)$  and a certain closed subspace of the Skorokhod space, denoted by  $\mathcal{D}[0, 1]$ . With the help of this isomorphism we obtain a stochastic functional equation satisfied by both the main eigenvalue and its associated eigenfunction. Then a representation of the main eigenvalue in terms of the Brownian motion and a limit of a quotient between linear combinations of log-normal random variables are also obtained. Section 5 is devoted to obtain upper and lower bounds for the expected value of the main eigenvalue and the main idea is the use of the reflection principle of the Brownian motion to overcome the complex

combinatorial problem arising from the representation obtained in Section 4. In Section 6 we use the results of Sections 2-5 to give a proof of existence and finiteness of the quenched topological pressure. In Section 7 we briefly discuss some natural extensions of our results, as to more general alphabets and to other random potentials. We also explain the difficulties in improving our estimates and obtaining the existence of the annealed pressure.

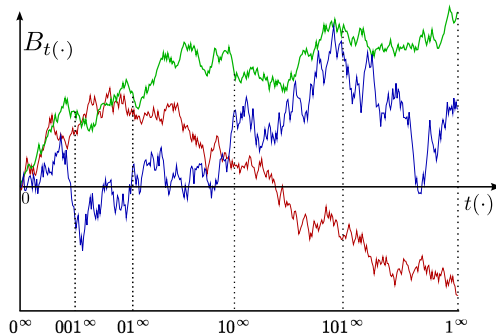
## 2 Brownian Type Potentials

Let  $\{B_t : t \in [0, 1]\}$  be the standard one-dimensional Brownian motion, i.e., a stochastic process with continuous paths almost surely, stationary independent increments and  $B_t \sim \mathcal{N}(0, t)$  for each  $t \geq 0$ , see [KS91, Lig10].

The idea is to use the almost certain continuity of its paths  $t \mapsto B_t$  to define an almost certain continuous random potential on the symbolic space  $\Omega = \{0, 1\}^{\mathbb{N}}$ .

The construction of this random potential is as follows. We consider the function (abusing notation)  $t : \Omega \rightarrow [0, 1]$  from the symbolic space to the closed unit interval  $[0, 1]$ , defined for each  $x \in \Omega$  by the expression  $t(x) = \sum_{n \geq 1} 2^{-n} x_n$ . The **Brownian potential** is the random mapping from  $\Omega$  to  $\mathbb{R}$  given by

$$\Omega \ni x \mapsto B_{t(x)}.$$



**Figure 1:** *Samples of the Brownian potential.*

$$a_1 \dots a_n^\infty := (a_1, \dots, a_n, a_n, a_n, \dots)$$

From the very definition of the Brownian motion follows that the Brownian potential is almost certain a continuous potential over the symbolic space  $\Omega$ . Therefore we have almost certain a well defined Ruelle operator  $\mathcal{L}_{B_{t(\cdot)}}$  sending  $C(\Omega)$  to itself. This “random” operator takes a continuous function

$\varphi$  to another continuous function  $\mathcal{L}_{B_{t(\cdot)}}(\varphi)$  that is given for each  $x \in \Omega$  by

$$\mathcal{L}_{B_{t(\cdot)}}(\varphi)(x) = \sum_{y \in \Omega: \sigma(y)=x} \exp(B_{t(y)})\varphi(y). \quad (1)$$

We observe that rhs of (1) let it clear that  $\mathcal{L}_{B_{t(\cdot)}}(\varphi)(x)$  is a random variable, for any choice of  $x \in \Omega$  and  $\varphi \in C(\Omega)$ .

**Proposition 1.** *The spectral radius  $\lambda_B$  of the random Ruelle operator  $\mathcal{L}_{B_{t(\cdot)}}$  is a non-negative random variable.*

*Proof.* Of course, the only issue here is about measurability and we show how to overcome this by using the Gelfand Formula and expression (1). Indeed, by applying iteratively the expression (1) one can see that  $\mathcal{L}_{B_{t(\cdot)}}^n(\varphi)(x)$  is a random variable for any choice of  $\varphi \in C(\Omega)$  and  $x \in \Omega$ . For any fixed  $n \in \mathbb{N}$  and  $\varphi \in C(\Omega)$  there exists, by the definition of the supremum, a sequence  $(x_k)$  in  $\Omega$  such that  $\|\mathcal{L}_{B_{t(\cdot)}}^n(\varphi)\|_\infty = \lim_{k \rightarrow \infty} |\mathcal{L}_{B_{t(\cdot)}}^n(\varphi)(x_k)|$  so proving that  $\|\mathcal{L}_{B_{t(\cdot)}}^n(\varphi)\|_\infty$  is a random variable. Since the Gelfand's formula ensures that the spectral radius of  $\mathcal{L}_{B_{t(\cdot)}}$  is given by

$$\lambda_B = \lim_{n \rightarrow \infty} (\sup\{\|\mathcal{L}_{B_{t(\cdot)}}^n(\varphi)\|_\infty : \varphi \in C(\Omega) \text{ and } \|\varphi\|_\infty = 1\})^{\frac{1}{n}}$$

we can argue similarly as above to conclude that  $\lambda_B$  is a random variable.  $\square$

**Proposition 2.** *The random variable  $\lambda_B$  is the main eigenvalue of the random Ruelle operator  $\mathcal{L}_{B_{t(\cdot)}}$  almost surely. Moreover this operator has a positive continuous eigenfunction  $h_B$  and, additionally,  $h_B(x)$  is a random variable, for each  $x \in \Omega$ .*

*Proof.* A typical trajectory of the Brownian motion in the unit interval  $[0, 1]$  is  $\gamma$ -Hölder continuous function for any  $\gamma < 1/2$ , see [KS91, Lig10]. A simple computation shows that  $|t(x) - t(y)| \leq d(x, y)$ , where  $d(x, y) = 2^{-N}$  and  $N = \inf\{n : x_j = y_j \text{ for } 1 \leq j < n \text{ and } x_n \neq y_n\}$ . Therefore almost certain we have

$$|B_{t(x)} - B_{t(y)}| \leq \text{const.} |t(x) - t(y)|^\gamma \leq \text{const.} (d(x, y))^\gamma.$$

So our random potentials  $x \mapsto B_{t(x)}$  are almost certain  $\gamma$ -Hölder continuous for any  $\gamma < 1/2$ . From the classical Ruelle-Perron-Fröbenius Theorem follows that  $\lambda_B$ , the random spectral radius of  $\mathcal{L}_{B_{t(\cdot)}}$ , is actually the main eigenvalue and it has geometric multiplicity one almost surely. From item iii) of Theorem A we have for each  $x \in \Omega$  that

$$\lambda_B^{-n} \cdot \mathcal{L}_{B_{t(\cdot)}}^n(1)(x) \rightarrow h(x).$$

Since the lhs above is a random variable it follows that  $h(x)$  is also a random variable.  $\square$

Before proceed we recall some elementary facts about dual of the Ruelle operator associated to a deterministic general continuous potential. As we mentioned before  $(\Omega, d)$  is a compact metric space so one can apply the Riesz-Markov Theorem to prove that  $C^*(\Omega)$  is isomorphic to  $\mathcal{M}_s(\Omega)$ , the space of all signed Radon measures. Therefore we can define  $\mathcal{L}_f^*$ , the dual of the Ruelle operator, as the unique continuous map from  $\mathcal{M}_s(\Omega)$  to itself satisfying for each  $\gamma \in \mathcal{M}_s(\Omega)$  the following identity

$$\int_{\Omega} \mathcal{L}_f(\varphi) d\gamma = \int_{\Omega} \varphi d(\mathcal{L}_f^* \gamma) \quad \forall \varphi \in C(\Omega). \quad (2)$$

From the positivity of  $\mathcal{L}_f$  follows that the map  $\gamma \mapsto \mathcal{L}_f^*(\gamma)/\mathcal{L}_f^*(\gamma)(1)$  sends the space of all Borel probability measures  $\mathcal{P}(\Omega)$  to itself. Since  $\mathcal{P}(\Omega)$  is convex set and compact in the weak topology (which is Hausdorff in this case) and the mapping  $\gamma \mapsto \mathcal{L}_f^*(\gamma)/\mathcal{L}_f^*(\gamma)(1)$  is continuous, the Tychonoff-Schauder Theorem ensures the existence of at least one Borel probability measure  $\nu_f$  such that  $\mathcal{L}_f^*(\nu_f) = \mathcal{L}_f^*(\nu_f)(1)\nu_f$ . The eigenvalue  $\lambda_f \equiv \mathcal{L}_f^*(\nu_f)(1)$  is positive and equals to the spectral radius of  $\mathcal{L}_f$  and therefore independent of the choice of the fixed point  $\nu_f$ , see [Bal00, Bow08, PP90].

If  $X$  is a continuous random potential, then the Ruelle operator  $\mathcal{L}_X$  sends  $C(\Omega)$  to itself almost surely. We claim that for any continuous random potential  $X$  the spectral radius of the operator  $\mathcal{L}_X$ , denoted by  $\lambda_X$  is a random variable. In fact, the claim follows from the arguments given in the proof of Proposition 1 replacing  $B_{t(\cdot)}$  by  $X$ .

**Definition 2** (Phase Transition). *Let  $X$  be a continuous random potential and  $\lambda_X$  the random spectral radius of  $\mathcal{L}_X$ . We say that the random potential  $X$  does not present phase transition if the eigenvalue problem  $\mathcal{L}_X^* \nu = \lambda_X \nu$  has a unique solution in  $\mathcal{P}(\Omega)$  almost surely. Otherwise, we say that  $X$  presents phase transition.*

**Corollary 1** (Absence of Phase Transition for Brownian Potentials). *Almost surely the dual of the Ruelle operator  $\mathcal{L}_{B_{t(\cdot)}}^*$  has one eigenprobability associated to  $\lambda_B$ .*

*Proof.* From the Theorem 1 follows that  $\lambda_B$  is both the spectral radius and the main eigenvalue of  $\mathcal{L}_{B_{t(\cdot)}}$ . So the corollary follows from the almost certain  $\gamma$ -Hölder continuity of the mapping  $\Omega \ni x \mapsto B_{t(x)}$ , obtained in the proof of Theorem 1, together with the item ii) of Theorem A.  $\square$

### 3 $C(\Omega) \simeq \mathcal{D}[0, 1]$

Let us denote by  $\mathcal{D}$  the subset of the unit interval  $[0, 1]$  such that all of its points have exactly two distinct binary expansion representations, i.e.,

$$\mathcal{D} \equiv \{s \in [0, 1] : \#t^{-1}(s) = 2\}.$$

If  $t(x) \in \mathcal{D}$  then there is a unique  $y \in \Omega \setminus \{x\}$  such that  $t(x) = t(y)$ . Moreover such  $x$  and  $y$  are always comparable in the lexicographic order. When  $x$  is smaller than  $y$  in this order we indicate this by using the notation  $x \prec_{\text{Lex}} y$ .

We now introduce a proper subset of the Skorokhod space that will play a fundamental role in this section. More details on the Skorokhod space can be found in [Bil99]. The functions belonging to the subspace mentioned above are càdlàg on the unit interval but only allowed to jump on the set  $\mathcal{D}$ . We use the convenient notation  $\mathcal{D}[0, 1]$  to denote this subspace and it is characterized as follows

$$\mathcal{D}[0, 1] \equiv \left\{ F : [0, 1] \rightarrow \mathbb{R} : \begin{array}{l} F \text{ is continuous in } [0, 1] \setminus \mathcal{D} \text{ and } \forall t \in \mathcal{D} \\ \text{the left limit } \lim_{s \uparrow t} F(s) \text{ exists;} \\ \text{the right limit } \lim_{s \downarrow t} F(s) = F(t). \end{array} \right\}.$$

We want to relate this space to  $C(\Omega)$  and to this end we introduce the following linear operator  $\Theta : \mathcal{D}[0, 1] \rightarrow C(\Omega)$  which sends a function  $F \in \mathcal{D}[0, 1]$  to a function  $f \in C(\Omega)$  whose its definition is

- $f(x) = F(t(x))$  if  $x \notin t^{-1}(\mathcal{D})$  and
- if  $x \in t^{-1}(\mathcal{D})$  then there exists a unique  $y \in \Omega \setminus \{x\}$  such that  $t(x) = t(y)$ . If  $x \prec_{\text{Lex}} y$  we put  $f(x) = \lim_{s \uparrow t(x)} F(s)$  and  $f(y) = \lim_{s \downarrow t(y)} F(s) = F(t(y))$ .

The linearity of  $\Theta$  is obvious, but the statement  $\Theta(F) \in C(\Omega)$  whenever  $F \in \mathcal{D}[0, 1]$  requires a proof and this is the content of the next proposition.

**Proposition 3.** *For every  $F \in \mathcal{D}[0, 1]$  we have that  $\Theta(F) \in C(\Omega)$ .*

*Proof.* Let  $x \in \Omega$  be an arbitrary point and  $F \in \mathcal{D}[0, 1]$ . In order to prove the continuity of  $\Theta(F)$  at  $x$  we split the analysis in two cases: whether  $F$  is continuous or not in  $t(x)$ . Suppose that  $F$  is continuous in  $t(x)$ . In this case independently whether  $t(x)$  belongs or not to  $\mathcal{D}$  we have  $\Theta(F)(x) = F(t(x))$ . Given  $\varepsilon > 0$  follows from the continuity of  $F$  that there is  $\delta > 0$  such that if  $|s - t(x)| < \delta$  then  $|F(s) - F(t(x))| < \varepsilon$ . From the continuity of  $t$  it is possible to find  $\eta > 0$  such that if  $d(y, x) < \eta$  then  $|t(x) - t(y)| < \delta$ .



Suppose by contradiction that  $\Theta(F)$  is not continuous at  $x$ . So there is a sequence  $y_n \in \Omega$  such that  $y_n \rightarrow x$  but  $|\Theta(F)(y_n) - \Theta(F)(x)| = |\Theta(F)(y_n) - F(t(x))| \geq \varepsilon$  for all  $n \in \mathbb{N}$ . Because of the previous observations for  $n$  large enough  $\Theta(F)(y_n) \neq F(t(y_n))$  otherwise we reach a contradiction. So the only possibility is  $\Theta(F)(y_n) = \lim_{s \uparrow t(y_n)} F(s)$  for all  $n \geq n_0$ . Again, we reach a contradiction since for large enough  $n$  we have  $t(y_n)$  close to  $t(x)$  and therefore by taking  $s$  close enough to  $t(x)$  we have  $F(s)$  close to  $F(t(x))$  which is a contradiction with  $|\Theta(F)(y_n) - F(t(x))| \geq \varepsilon$ .

Now we assume that  $F$  has a jump at  $t(x)$ . In this case by the definition of  $\mathcal{D}[0, 1]$  there exist exactly one point  $y \in \Omega \setminus \{x\}$  such that  $t(x) = t(y)$ . We give the argument for  $x \prec_{\text{Lex}} y$  on the other case the analysis is similar. Note that

$$\begin{aligned} y &= (x_1, \dots, x_n, 1, 0, 0, \dots) \\ x &= (x_1, \dots, x_n, 0, 1, 1, \dots). \end{aligned}$$

From the definition we have

$$\Theta(F)(x) = \lim_{s \uparrow t(x)} F(s).$$

Given  $\varepsilon > 0$  there is  $\delta > 0$  such that for all  $s$  satisfying  $t(x) - \delta < s \leq t(x)$  we have  $|F(s) - \Theta(F)(x)| < \varepsilon$ . Note that for every  $z \in \Omega$  such that  $d(z, x) \leq 2^{n+1}$  we have  $t(z) \leq t(x)$ . By taking  $z$  close enough to  $x$  and proceed similarly to the previous case one can show that  $\Theta(F)(z)$  is close to  $\Theta(F)(x)$ , thus proving the continuity of  $\Theta(F)$  at  $x$ . Since  $x$  is arbitrary the proof is complete.  $\square$

**Theorem 1.** *The operator  $\Theta$  is a Banach isomorphism (linear isometry) between the Banach spaces  $(\mathcal{D}[0, 1], \|\cdot\|_\infty)$  and  $(C(\Omega), \|\cdot\|_\infty)$ .*

*Proof.* We first construct the inverse operator  $\Theta^{-1}$  and then we proceed to show that  $\Theta$  is an isometry.

Let  $f \in C(\Omega)$  be an arbitrary continuous function. If  $s \in [0, 1] \setminus \mathcal{D}$  we define  $F(s) = f(t^{-1}(s))$ . If  $s \in \mathcal{D}$  then  $t^{-1}(s) = \{x, y\}$  with  $x \neq y$ . If  $x \prec_{\text{Lex}} y$  we put  $F(s) = f(y)$ . We claim that  $F \in \mathcal{D}[0, 1]$ . Indeed, if  $s \in [0, 1] \setminus \mathcal{D}$ , then  $s$  has a unique binary expansion  $t^{-1}(s) = (x_1, \dots, x_n, \dots)$ . For each  $n \in \mathbb{N}$  and  $j = 1, \dots, 2^n$  we consider the following collection of open intervals

$$I_n^j = \left( \frac{j-1}{2^n}, \frac{j}{2^n} \right)$$

Note that  $\mathcal{D} \cup \{0, 1\} = \bigcup_{n \geq 1} \bigcup_{j=1}^{2^n} \partial I_n^j$ . Note that if  $s \in [0, 1] \setminus \mathcal{D}$ , then  $s$  belongs to the interior of infinitely many  $I_n^j$ 's. Given  $\varepsilon > 0$  there is  $\delta > 0$  such that for all  $y \in \Omega$  satisfying  $d(t^{-1}(s), y) < \delta$  we have  $|f(t^{-1}(s)) - f(y)| < \varepsilon$ . As mentioned before we can choose  $n$  as large as we please so that  $s \in I_n^j$ ,

for some  $j = 1, \dots, 2^n$ . From the definition of  $I_n^j$  the first  $n$  terms of the binary expansion of any  $r \in I_n^j$  coincides with  $x_1, \dots, x_n$ . For  $n$  such that  $2^{-n} < \delta$  we have  $d(t^{-1}(s), t^{-1}(r)) < \delta$  and therefore for all  $r \in I_n^j$  we have  $|F(r) - F(s)| \leq \max_{z \in t^{-1}(r)} |f(z) - f(x)| < \varepsilon$ .

Suppose that  $t(x) \in \mathcal{D}$  and  $\{x, y\} = t^{-1}(x)$ . Without loss of generality we can assume that  $x \prec_{\text{Lex}} y$ . For  $n \geq n_0$  we have  $t(x) \in \partial I_n^j \cap \partial I_n^{j+1}$  for a unique value of  $j \in \{1, \dots, 2^n\}$ . Every element in  $t^{-1}(I_n^j)$  has its first  $n$  digits in the binary expansion equals to  $x_1, x_2, \dots, x_n$ . Therefore some small neighborhood of  $x$  is sent by  $t$  in  $I_n^j \cup \{t(x)\}$ . From the continuity of  $f$  at  $x$  it follows that the small neighborhood defined above is sent in a  $\varepsilon$  neighborhood of  $f(x)$  which proves the existence of the left limit of  $F$  at  $t(x)$ . The right continuity of  $F$  at  $t(x) \in \mathcal{D}$  is proved in similar way.

Notice that the argument in the two previous paragraphs also proves that the mapping

$$\begin{aligned} C(\Omega) \ni f &\mapsto F \\ s &\mapsto \begin{cases} f(t^{-1}(s)), & \text{if } s \in [0, 1] \setminus \mathcal{D}; \\ f(y), & \text{if } t^{-1}(s) = \{x, y\} \text{ and } x \prec_{\text{Lex}} y. \end{cases} \end{aligned}$$

is a left inverse of  $\Theta$ . Of course,  $\Theta$  is right inverse of this operator so  $\Theta$  is a bijection.

Remains to prove that  $\Theta$  is an isometry. From its very definition we have

$$\|\Theta(F)\|_\infty = \sup_{s \in [0, 1]} |\Theta(F)(s)| \leq \|F\|_\infty.$$

If the supremum above is attained at certain point  $s \in [0, 1] \setminus \mathcal{D}$ , take  $x \in \Omega$  such that  $t(x) = s$ , then we have  $\|F\|_\infty = |F(s)| = |\Theta(F)(x)| \leq \|\Theta(F)\|_\infty$ . If the supremum is not attained for some  $s \in \mathcal{D}$  the conclusion is analogous, otherwise  $\|F\|_\infty = \lim_{n \rightarrow \infty} \lim_{r \uparrow s_n} |F(r)| = \lim_{n \rightarrow \infty} |\Theta(F)(x_n)|$ , where  $x_n$  is the smallest point, with respect to the lexicographic order, in  $t^{-1}(s_n)$  and the proposition follows.  $\square$

**Remark 1** (Wiener Spaces). *Recall that the classical Wiener space on the closed unit interval  $[0, 1]$ , notation  $W[0, 1]$ , is the set of all real continuous functions  $F : [0, 1] \rightarrow \mathbb{R}$  such that  $F(0) = 0$ . Consider the closed subspace of  $C(\Omega)$  given by*

$$W(\Omega) \equiv \{f \in C(\Omega) : f(0^\infty) = 0, \ f(x) = f(y) \text{ whenever } t(x) = t(y)\},$$

where the mapping  $t : \Omega \rightarrow \mathbb{R}$  was previously defined. It is easy to see that for every  $F \in W[0, 1]$  its composition with  $t$  give us a mapping  $F \circ t \in W(\Omega)$ . Moreover the linear mapping  $F \mapsto F \circ t$  is an isometry from  $W[0, 1]$  to  $W(\Omega)$ .

## 4 Stochastic Functional Equation for $\lambda_B$

In this section we apply the results above obtained. The idea is to obtain a stochastic functional equation using the operator  $\Theta$  and Proposition 2 to semi-explicit representation of the main eigenvalue  $\lambda_B$  of the random Ruelle operator  $\mathcal{L}_{B_{t(\cdot)}}$ .

Since  $h_B$  is a continuous function almost surely, we can associate to it a càdlàg process  $\{X_s : 0 \leq s \leq 1\}$  so that  $\Theta(X_{(\cdot)}) = h_B$ . By simple algebraic manipulation we can see that  $t(0x) = t(x)/2$  and  $t(1x) = 1/2 + t(x)/2$ . By applying the inverse of the operator  $\Theta$  on both sides of the above equation we get the following *stochastic functional equation*

$$\exp(B_{\frac{t}{2}})X_{\frac{t}{2}} + \exp(B_{\frac{1}{2}+\frac{t}{2}})X_{\frac{1}{2}+\frac{t}{2}} = \lambda_B X_t.$$

Now we have to play with the properties of the Brownian motion and the Ruelle operator to obtain the law of the main eigenvalue. Recalling that  $B_0 = 0$  almost surely, so we get from the stochastic functional equation that

$$X_0 + \exp(B_{\frac{1}{2}})X_{\frac{1}{2}} = \lambda_B X_0.$$

Therefore we get by isolating the eigenvalue (recall that  $X_s > 0$  for each  $s \in [0, 1]$ ) the following equality a.s.

$$\lambda_B = 1 + \exp(B_{\frac{1}{2}}) \frac{X_{\frac{1}{2}}}{X_0} \tag{3}$$

By using again the operator  $\Theta$  one can see that  $X_0 = h_B(0, 0, \dots)$  and  $X_{1/2} = h_B(1, 0, 0, \dots)$ , recall that  $(0, 1, 1, \dots) \prec_{\text{Lex}} (1, 0, 0, \dots)$ . From the Theorem A we have almost certain

$$\frac{X_{\frac{1}{2}}}{X_0} = \frac{h_B(1, 0, 0, \dots)}{h_B(0, 0, \dots)} = \lim_{n \rightarrow \infty} \frac{\mathcal{L}_{B_{t(\cdot)}}^n(1)(1, 0, 0, \dots)}{\mathcal{L}_{B_{t(\cdot)}}^n(1)(0, 0, 0, \dots)}.$$

From where we conclude that

$$\lambda_B = 1 + \exp(B_{\frac{1}{2}}) \lim_{n \rightarrow \infty} \frac{\mathcal{L}_{B_{t(\cdot)}}^n(1)(1, 0, 0, \dots)}{\mathcal{L}_{B_{t(\cdot)}}^n(1)(0, 0, 0, \dots)}.$$

## 5 Bounds on the Expected Value of $\lambda_B$

In the deterministic case, when the potential is assumed to have Hölder, Walters or Bowen regularity, it is easy to obtain lower and upper bounds of

the main eigenvalue of the Ruelle operator by using the supremum norm of the potential. We recall that when the potential have such regularity the main eigenvalue do exists and is actually equals to the spectral radius.

Since here we are considering random potentials lower and upper bounds for the expected values, with respect to the Wiener measure, of the spectral radius is a natural question to ask.

Opposed to the deterministic case the supremum norm of  $B_{t(\cdot)}$  by itself does not help in finding bounds to the spectral radius. The reason is  $\sup\{|B_{t(x)}| : x \in \Omega\}$  is an unbounded random variable. The representation of  $\lambda_B$  obtained in the last section is not suitable to take expectations and very hard to bound due its complex combinatorial nature. Therefore a different approach is needed to bound the expected value of main eigenvalue and the idea is based on the reflection principle of the Brownian motion.

**Theorem 2.** *Let  $\lambda_B$  be the random variable defining the spectral radius of the random Ruelle operator  $\mathcal{L}_{B_{t(\cdot)}}$ . Then  $\lambda_B$  has finite first moment, with respect to the Wiener measure, and moreover*

$$\exp(1/2) \leq \mathbb{E}[\lambda_B] \leq 4 \exp(1/2).$$

*Proof.* We begin with the lower bound. To lighten the notation we use the symbol  $1^\infty$  to denote the constant sequence  $(1, 1, 1, \dots)$ . By the definition of the supremum we have

$$\begin{aligned} \sup_{\varphi: \|\varphi\|_\infty=1} \|\mathcal{L}_{B_{t(\cdot)}}^n(\varphi)\|_\infty &= \sup_{\varphi: \|\varphi\|_\infty=1} \sup_{x \in \Omega} |\mathcal{L}_{B_{t(\cdot)}}^n(\varphi)(x)| \\ &= \sup_{\varphi: \|\varphi\|_\infty=1} \sup_{x \in \Omega} \left| \sum_{a_1, \dots, a_n} \exp\left(\sum_{j=0}^{n-1} B_{t(\sigma^j(a_1 \dots a_n x))}\right) \varphi(\sigma^j(a_1 \dots a_n x)) \right| \\ &\geq \exp \sum_{j=0}^{n-1} B_{t(1^\infty)} \\ &\geq \exp(nB_1). \end{aligned}$$

By taking the  $n$ -th root, the limit when  $n \rightarrow \infty$  and the expectation in the above inequality we get that

$$\mathbb{E}[\lambda_B] \geq \mathbb{E}[\exp(B_1)] = \exp(1/2).$$

To get the upper bound we will take advantage of the reflection principle

of the Wiener process. We first observe that

$$\begin{aligned}
\sup_{\varphi: \|\varphi\|_\infty=1} \|\mathcal{L}_{B_t(\cdot)}^n(\varphi)\|_\infty &\leq \sup_{x \in \Omega} |\mathcal{L}_{B_t(\cdot)}^n(1)(x)| \\
&= \sup_{x \in \Omega} \sum_{a_1, \dots, a_n} \exp \sum_{j=0}^{n-1} B_{t(\sigma^j(a_1 \dots a_n x))} \\
&\leq \sum_{a_1, \dots, a_n} \sup_{x \in \Omega} \exp \sum_{j=0}^{n-1} B_{t(\sigma^j(a_1 \dots a_n x))} \\
&\leq 2^n \exp(n \max\{B_t : 0 \leq t \leq 1\}).
\end{aligned}$$

By taking the  $n$ -th root and the limit when  $n \rightarrow \infty$  in the last inequality we get  $\lambda_B \leq 2 \exp(M_1)$ , where  $M_1 \equiv \max\{B_t : 0 \leq t \leq 1\}$ . From the reflection principle of the Wiener process it follows that

$$\mathbb{E}[\lambda_B] \leq 2\mathbb{E}[\exp(M_1)] \leq 4 \cdot \mathbb{E}[\exp B_1] = 4 \exp(1/2). \quad \square$$

## 6 Existence and Finiteness of the Quenched Pressure

By using the  $\gamma$ -Hölder regularity of the Brownian potential  $x \mapsto B_{t(x)}$  and a classical result of Thermodynamic formalism, we have almost certain

$$\log \lambda_B = P(B_{t(\cdot)}) = \sup_{\mu \in \mathcal{P}_\sigma(\Omega)} \left\{ h(\mu) + \int_\Omega B_{t(\cdot)} d\mu \right\},$$

where  $P(\cdot)$  is the topological pressure,  $\mathcal{P}_\sigma(\Omega)$  is the set of all Borel probability measures that are shift invariant and  $h$  is the Kolmogorov-Sinai entropy. From (3) we know that  $\lambda_B = 1 + \exp(B_{1/2})(X_{1/2}/X_0)$ , and  $X_{1/2}/X_0 \geq 0$ . Therefore we can conclude that  $\log \lambda_B$  is non-negative random variable and its expected value with respect to the Wiener measure is well defined, at least as an extended real number.

Finally, by using the elementary inequality  $\log \lambda_B \leq \lambda_B$  and the Theorem 2 we have that  $\mathbb{E}[\log \lambda_B] < +\infty$ , thus proving the finiteness of the quenched pressure. Using once more the  $\gamma$ -Hölder regularity of the Brownian potential we have almost certain that

$$\log \lambda_B = \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{L}_{B_{t(\cdot)}}^n(1)(\sigma^n(x)), \quad \forall x \in \Omega$$

and therefore

$$P^{\text{quenched}}(B_{t(\cdot)}) \equiv \mathbb{E}[\log \lambda_B] = \mathbb{E} \left[ \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{L}_{B_{t(\cdot)}}^n(1)(\sigma^n(x)) \right].$$

Finally, by the Theorem 2 and the Jensen Inequality we get

$$0 \leq P^{\text{quenched}}(B_{t(\cdot)}) = \mathbb{E}[\log \lambda_B] \leq \log \mathbb{E}[\lambda_B] \leq \log 4 + \frac{1}{2}.$$

## 7 Concluding Remarks

As mentioned, all the results presented in this paper can be extended to the space  $\Omega = \{0, \dots, m-1\}^{\mathbb{N}}$ . For such state spaces the map  $t : \Omega \rightarrow [0, 1]$  is given by  $(x_1, x_2, \dots) \mapsto \sum_{n \geq 1} m^{-n} x_n$ . Analogous arguments can be used to prove the Proposition 1 and Proposition 2. In the Section 3, the arguments would be completely analogous, only changing appropriately the set  $\mathcal{D}$ .

In the Section 4, we would have a subtle change in the stochastic functional equation. Now we have for every  $a \in \{0, \dots, m-1\}$  and  $x \in \Omega$ ,  $t(ax) = a/m + t(x)/m$ , so a slightly different stochastic functional equation shows up  $\exp(B_{t/m})X_{t/m} + \exp(B_{1/m+t/m})X_{1/m+t/m} = \lambda_B X_t$ . Thus, by taking  $t = 0$ , we recover the representation of the main eigenvalue

$$\lambda_B = 1 + \exp(B_{\frac{1}{m}}) \lim_{n \rightarrow \infty} \frac{\mathcal{L}_{B_{t(\cdot)}}^n(1)(1, 0, 0, \dots)}{\mathcal{L}_{B_{t(\cdot)}}^n(1)(0, 0, 0, \dots)}.$$

The technique employed in Theorem 2 provided the following bounds

$$\exp(1/2) \leq \mathbb{E}[\lambda_B] \leq 2m \exp(1/2).$$

Therefore, by using the same arguments, one can also prove the existence and finiteness of the Quenched Pressure on this case.

Another natural question is whether we could study another random potential defined by a continuous time stochastic process other than a Brownian Motion. Note that the only property of the Brownian Motion used to guarantee the existence of the main eigenvalue is the almost certain Hölder continuity of the paths (in fact this is required only in  $[0, 1] \setminus \mathcal{D}$ ), which is not an exclusive property of Brownian Motion. In the functional stochastic equation, we only used that  $B_0 = 0$  almost surely, but for the general case all that is required is to add a constant to the potential. Here we used the reflection principle and the stationarity of the increments to bound  $\mathbb{E}[\lambda_B]$  which also follow for some other processes. Nevertheless if one has another technique to guarantee its finiteness, the general sequence described in the article still holds and the existence and finiteness of the Quenched Pressure is still guaranteed. Therefore, the natural candidate to extend the results is Fractional Brownian motion, which indeed leads to very analogous results.

The obstacle one faces to obtain tighter lower bounds for  $\mathbb{E}[\lambda_B]$  is of combinatorial nature and the complicated sums of dependents log-normal random variables that appear in the calculations and have to be controlled. Some informal computations suggest that the annealed pressure is also well defined for Brownian type potential and possibly given by  $\log(\text{ess sup } \lambda_B)$ , but unfortunately the techniques developed here seem not so helpful in attacking this problem.

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